Computing polynomials for sequences using the analysis of differences

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Abstract

This paper gives a method of computing polynomials for sequences by analysing the differences within the sequence. Rigorous justification for the method is provided along with an example.

Contents

1 Introduction

Given a sequence such as $5, 10, 25, ...$ which polynomials generate this sequence of numbers? Consider $f(x) = 5x^2 - 10x + 10$ then $f(1) = 5, f(2) = 10, f(3) = 25$ so f can be said to generate this sequence. Given this, does that mean $f(4) = 50$ is the next number in the sequence?

Consider the polynomial $g(x) = x^3 - x^2 + x + 4$ then $g(1) = 5, g(2) = 10, g(3) = 25$ so g can be said to generate this sequence also. However $g(4) = 56$. So what is the next number in the sequence? Is it 50 or 56 or something completely different?

Well the question is too ambiguous. Above are two examples of polynomials which generate the start of the sequence but differ on the fourth term. In actual fact you can pick any number you want to be the fourth term and there would exist a polynomial that generates the sequence $5, 10, 25$ and the fourth term would be what ever number you like.

In this article, I am going to present a simple method for computing these polynomials, which hides the complex methods of solving systems of linear equations taught in higher education Maths courses. Rigorous justifications for the method will also be provided.

By the end of this article you should be able to calculate a polynomial which generates the sequence $1, 1, 1, n$ where n can be any number you wish.

2 Differences

Definition 2.1. *For this article we define* $f_i := x^i$.

The above definition is used to make the article more easily readable.

Definition 2.2. *We define the first order difference of the polynomial* f *at a point* x *to be* $f(x+1) - f(x)$. *We denote the first order difference as* $D_1 f(x)$ *so we have:*

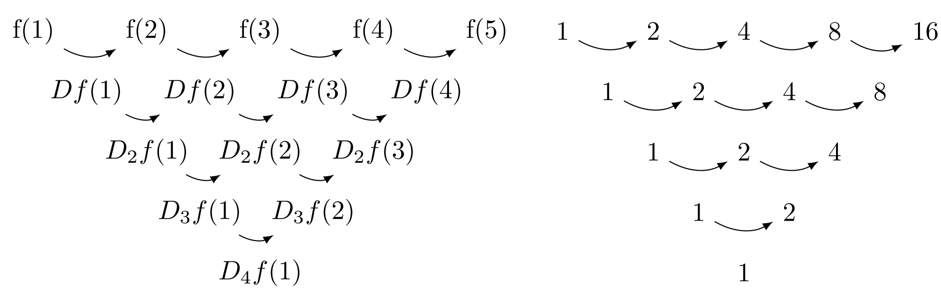
$$D_1 f(x) := f(x+1) - f(x)$$

Definition 2.3. *We denote the i-th order difference of the polynomial* f *at a point* x *to be* $D_i f(x)$. *We define the i-th order difference as*

$$D_i f(x) := D_{i-1} f(x+1) - D_{i-1} f(x)$$

For the first order difference $D_1 f(x)$ it is acceptable to drop the 1 and just denote it as $Df(x)$. We also define $D_0 f(x)$ to be $f(x)$.

The following diagram demonstrates the concept of n-th order differences.



It is worth noting that $D(D_n f)(x) = D_{n+1} f(x)$. This can be seen by letting $g(x) = D_n f(x)$ and then writing out $Dg(x)$ from the definition above.

Lemma 2.1. *Let* $g_1, g_2, ..., g_n$ *be* n *polynomials and* $\alpha_1, \alpha_2, ..., \alpha_n$ *be real numbers and let* f *be the polynomial* $f(x) := \sum_{i=1}^{n} \alpha_i g_i$ *then*

$$Df(x) = \sum_{i=1}^{n} \alpha_i D g_i(x)$$

Proof.

$$\begin{aligned} Df(x) &= f(x+1) - f(x) \\ &= \sum_{i=1}^{n} \alpha_i g_i(x+1) - \sum_{i=1}^{n} \alpha_i g_i(x) \\ &= \sum_{i=1}^{n} \left(\alpha_i g_i(x+1) - \alpha_i g_i(x)\right) \\ &= \sum_{i=1}^{n} \alpha_i \left(g_i(x+1) - g_i(x)\right) \\ &= \sum_{i=1}^{n} \alpha_i D g_i(x) \end{aligned}$$

□

Corollary 2.1.1. *Let $g_1, g_2, ..., g_n$ be n polynomials and $\alpha_1, \alpha_2, ..., \alpha_n$ be real numbers and let f be the polynomial $f(x) := \sum_{i=1}^{n} \alpha_i g_i$ and let j be a positive integer then*

$$D_j f(x) = \sum_{i=1}^{n} \alpha_i D_j g_i(x)$$

Proof. We prove the corollary by induction. For $j = 1$ we know this is true from lemma 2.1.

Assume this is true for all integers less than j then

$$\begin{aligned} D_j f(x) &= D_{j-1} f(x+1) - D_{j-1} f(x) \\ &= \sum_{i=1}^{n} \alpha_i D_{j-1} g_i(x+1) - \sum_{i=1}^{n} \alpha_i D_{j-1} g_i(x) \textit{(by the induction hypothesis)} \\ &= \sum_{i=1}^{n} \left(\alpha_i D_{j-1} g_i(x+1) - \alpha_i D_{j-1} g_i(x)\right) \\ &= \sum_{i=1}^{n} \alpha_i \left(D_{j-1} g_i(x+1) - D_{j-1} g_i(x)\right) \\ &= \sum_{i=1}^{n} \alpha_i D_j g_i(x) \end{aligned}$$

□

The above corollary just confirms that the n-th order difference operator is linear. That is to say, that if we have a polynomial $2x^3 + 7x$, then applying the difference operator on the whole polynomial would be the same as applying the difference operator separately to $2x^3$ and $7x$ and adding the results together.

We look at n-th order differences for a range of polynomials below and summarise the results in a table. We analyse one of these polynomials, namely

$f(x) = 2x^3 - x + 1$, below. We use the sequence generated by computing $f(1), f(2), f(3), f(4), f(5)$ and $f(6)$ in each case.

2		15		52		125		246		427
	13		37		73		121		181	
		24		36		48		60		
			12		12		12			
				0		0				

The following table has the results of performing the above analysis on a number of polynomials. As it only makes sense to evaluate $D_n f$ at a specific point, we will put a * for values of $D_n f$ where the value changes depending on x. If it is constant then we record the value.

Polynomial	Df	$D_2 f$	$D_3 f$	$D_4 f$	$D_5 f$
$2x^3 - x + 1$	*	*	12	0	0
$2x^3 - 5x^2 - 7x + 10$	*	*	12	0	0
$2x^3$	*	*	12	0	0
$x^3 + x^2 - x + 1$	*	*	6	0	0
$x^3 + 75x^2 - 10$	*	*	6	0	0
x^3	*	*	6	0	0
$5x + 6$	5	0	0	0	0
$5x + 1$	5	0	0	0	0
$3x + 9$	3	0	0	0	0
$3x + 2$	3	0	0	0	0
$x^2 - x + 1$	*	2	0	0	0
$x^2 + 5x + 16$	*	2	0	0	0
$3x^2 + 9x + 23$	*	6	0	0	0
$3x^2 - 7x + 12$	*	6	0	0	0
$x^4 + x^3 + x^2 + x + 1$	*	*	*	24	0

Looking at the table above there are some patterns that might be worth investigating. First of all it is clear that we eventually get a non-zero constant difference followed by 0 for successive differences. It is clear that once we get a constant value difference the successive differences will be 0 as these successive differences are defined by the previous differences and the differences between a constant sequence is 0.

Looking at polynomials of the same degree we see that we get the non-zero constant value at the same n-th order difference. That is for cubics (degree 3) we seem to get a non-zero constant value for $D_3 f$. For quadratics (degree 2) the non-zero constant appears at $D_2 f$ and for linear polynomials (degree 1) the non-zero constant value appears at $D_1 f$. The only quartic (degree 4) polynomial analysed has the non-zero constant value at $D_4 f$.

It would appear that for a polynomial of degree n that $D_n f$ is a non-zero constant value and that $D_j f = 0$ for $j > n$.

(1)

If we look at polynomials of the same degree and with the same coefficient of the highest term then it also looks like they have the same non-zero constant value. For example $2x^3 - x + 1$, $2x^3 - 5x^2 - 7x + 10$ and $2x^3$ all have $D_3 f = 12$. For $x^3 + x^2 - x + 1$, $x^3 + 75x^2 - 10$ and x^3 we have $D_3 f = 6$. It appears for polynomials of degree 3 that the value of $D_3 f$ is equal to 6 times the coefficient of the highest term.

If we look at the polynomials of degree 2 we see a similar pattern that $D_2 f$ is equal to 2 times the coefficient of the highest term and similarly for the polynomials of degree 1 we have D_f is 1 times the coefficient of the highest term. Finally looking at our only polynomial of degree 4 we have that $D_4 f$ is 24 times the coefficient of the highest term.

The numbers 1, 2, 6, 24 are interesting numbers to appear as $1 = 1$, $2 = 1 \times 2$, $6 = 1 \times 2 \times 3$ and $24 = 1 \times 2 \times 3 \times 4$. We have a notational shortcut for these numbers that are the product of consecutive numbers. We call them factorials. We write 1! to be 1, 2! to be 1×2, 3! to be $1 \times 2 \times 3$ and $n!$ to be $1 \times 2 \times 3 \times ... \times (n-1) \times n$.

If we summarise our results above using the factorial notation, with c representing the coefficient of the highest degree term in the polynomial, then we have:

Polynomials of degree 1 have $D_1 f$ equal to $1! \times c$

Polynomials of degree 2 have $D_2 f$ equal to $2! \times c$

Polynomials of degree 3 have $D_3 f$ equal to $3! \times c$

Polynomials of degree 4 have $D_4 f$ equal to $4! \times c$

It would appear that for a polynomial of degree n, with the coefficient of the highest term equal to c that $D_n f$ is a non-zero constant value and that $D_n f = n!c$.

(2)

Assuming statements 1 and 2 to be true we will go through an example of how to calculate a polynomial that generates a sequence. We will later try to prove statements 1 and 2.

Example

Let $2, 9, 22, 41, 66$ be the start of a sequence. Let us assume that this sequence is generate by a unknown polynomial that we will refer to as $s(x)$. That means our sequence is the sequence $s(1), s(2), s(3), s(4)$ and $s(5)$.

We do not know what degree $s(x)$ is at the moment. We will start by analysing the differences until we get 0's.

```
2     9     22    41    66
   7     13    19    25
      6     6     6
         0     0
            0
```

We see that the second order difference, $D_2 s$, is 6 and the higher order differences are 0. This would imply that our sequence is generated by a polynomial of degree 2 and that it's highest order coefficient is $\frac{6}{2!} = 3$.

So we know that $s(x) = 3x^2 + ax + b$ where a and b are unknown at the moment.

Now define $s_1(x) := s(x) - 3x^2$. Since $s(x) = 3x^2 + ax + b$, this would mean that $s_1(x) = ax + b$. $s_1(x)$ is a polynomial of lower degree than s. If we could generate the sequence $s_1(1), s_1(2), s_1(3), s_1(4), s_1(5)$ then we could analyse the differences of this sequence to work out a, which gives us more information about $s(x)$.

How can we calculate the sequence $s_1(1), s_1(2), s_1(3), s_1(4), s_1(5)$?

That's easy. From the definition of $s_1(x)$ we have $s_1(1) = s(1) - 3 \times (1)^2$. We know $s(1)$ as this is the first number in our original sequence! Similarly for $s_1(2), s_1(3), s_1(4)$ and $s_1(5)$.

This new sequence generated by $s_1(x)$ is calculated to be $2 - 3 \times 1^2, 9 - 3 \times 2^2, 22 - 3 \times 3^2, 41 - 3 \times 4^2, 66 - 3 \times 5^2$. This new sequence is $-1, -3, -5, -7, -9$.

Let's analyse the differences in this sequence.

```
-1    -3    -5    -7    -9
   -2    -2    -2    -2
      0     0     0
         0     0
            0
```

We see that the first order difference, $D_1 s_1$, is -2 and the higher order differences are 0. This would imply that our sequence is generated by a polynomial of degree 1 (as expected) and that it's highest order coefficient is $\frac{-2}{1!} = -2$.

So we know that $s_1(x) = -2x + b$ where b is unknown at the moment. This also means that we now know a so we have more information on $s(x)$. So far we have $s(x) = 3x^2 - 2x + b$. We still need to work out b.

If we repeat the procedure above and define $s_2(x) := s_1(x) - (-2x)$. From the definition of $s_1(x)$ we know that $s_2(x) = b$. Can we calculate the sequence $s_2(1), s_2(2), s_2(3), s_2(4), s_2(5)$. Well yes, we know $s_1(x)$ for $x = 1, 2, 3, 4, 5$ as that was our second sequence we calculated. So we can calculate the sequence $s_2(1), s_2(2), s_2(3), s_2(4), s_2(5)$ as $-1 + 2 \times 1, -3 + 2 \times 2, -5 + 2 \times 3, -7 + 2 \times 4, -9 + 2 \times 5$ so we have the sequence $1, 1, 1, 1, 1$. This means that our b is 1.

This leaves us with $s(x) = 3x^2 - 2x + 1$. We can plug in the values 1, 2, 3, 4 and 5 to check that it generates 2, 9, 22, 41, 66.

We will formalise this method below.

Method

Assume we have a sequence of numbers $a_{1,1}, a_{1,2}, a_{1,3}, ..., a_{1,n}$. The first subscript number just represents the iteration of the following instructions. At the start we are on the 1^{st} iteration and we will use the variable i to refer to the iteration. So $i = 1$ to start off with.

1. Analyse differences of the sequence $a_{i,1}, a_{i,2}, ..., a_{i,n}$ until we have that the ${j_i}^{\text{th}}$-order difference is a non-zero constant for some j_i. Let c_i be the non-zero constant.
2. Define a polynomial $s_i(x) = \frac{c_i}{j_i!} x^{j_i}$.
3. Calculate a new sequence $a_{i+1,1}, a_{i+1,2}, ..., a_{i+1,n}$ by defining $a_{i+1,d} = a_{i,d} - s_i(d)$ for $d = 1, 2, ..., n$.
4. If the sequence $a_{i+1,1}, a_{i+1,2}, ..., a_{i+1,n}$ is all 0 then stop.
5. Go to step 1 and increase the iteration number, i, by 1.

When this procedure terminates we would have defined polynomials $s_1(x), s_2(x), ..., s_i(x)$ (one for each iteration of the instructions).

Define $s(x) = \sum_{k=1}^{i} s_k(x) = s_1(x) + s_2(x) + ... + s_i(x)$ then this polynomial $s(x)$ is a polynomial that generates our original sequence $a_{i,1}, a_{i,2}, ..., a_{i,n}$

Justification

To see this, we just need to show that $s(i) = a_{1,i}$ for $i = 1, 2, ..., n$.

To show this we will unwind the sequences we have generated during the procedure.

On the last iteration we ended up with a sequence of all 0's. So for $i = 1, 2, ..., n$ we have (assuming j is the last iteration):

$$
\begin{aligned}
0 &= a_{j,i} \\
&= a_{j-1,i} - s_{j-1}(i) \\
&= a_{j-2,i} - s_{j-2}(i) - s_{j-1}(i) \\
&\vdots \qquad = a_{1,i} - s_1(i) - s_2(i) - ... - s_{j-2}(i) - s_{j-1}(i) \\
&= a_{1,i} - (s_1(i) + s_2(i) + ... + s_{j-2}(i) + s_{j-1}(i)) \\
&= a_{1,i} - s(i)
\end{aligned}
$$

Since $a_{1,i} - s(i) = 0$ for all $i = 1, 2, ..., n$ it follows that $s(i) = a_{1,i}$ for all $i = 1, 2, ..., n$. This means that our polynomial $s(x)$ does generate the sequence.

However, we haven't shown that this procedure will terminate. i.e. We will eventually get a sequence of all zeros in an iteration and this is crucial for rigorous justification.

We know that we can find a polynomial of degree n that generates a sequence of length $n+1$ as we can solve the system of linear equations of $n+1$ unknowns with $n+1$ equations.

That is to say given a sequence $a_1, a_2, ..., a_n, a_{n+1}$ then we could write a polynomial that generates this sequence as $s(x) = c_{n+1}x^n + c_n x^{n-1} + ... + c_3x^2 + c_2x + c_1$ then generate $n+1$ equations by plugging in the values of the sequence.

$$
\begin{aligned}
s(1) &= a_1 \\
s(2) &= a_2 \\
&\vdots \\
s(n) &= a_n \\
s(n+1) &= a_{n+1}
\end{aligned}
$$

If we have a sequence of length $n+1$ that can be generated by a polynomial with degree less than n then you will find that n-th order difference will be 0 and will be a non-zero constant for a lower order difference so without loss of generality assume we have a sequence that is generated by a polynomial $s(x) = c_{n+1}x^n + c_n x^{n-1} + ... + c_3x^2 + c_2x + c_1$ and that $c_{n+1} \neq 0$.

Then by statement 2, in our procedure we will define $s_1(x)$ to be $s_1(x) = c_{n+1}x^n$ and so when we generate the next sequence we will be generating a sequence which is generated by $s(x) - s_1(x)$, let's call it $n(x)$. By definition we would have $n(x) = s(x) - s_1(x) = \left(c_{n+1}x^n + c_n x^{n-1} + ... + c_3x^2 + c_2x + c_1\right) - c_{n+1}x^n = c_n x^{n-1} + ... + c_3x^2 + c_2x + c_1$. So $n(x)$ is a polynomial of degree strictly less than the degree of $s(x)$.

This means that in each iteration we are generating a new sequence which is generated by a polynomial of degree strictly less than the last. Because of this we will eventually end up with a sequence which is generated by a degree 0 polynomial (a constant value). At this stage the iteration will generate a new sequence which is equal the constant value minus itself which will be a 0. This means that the procedure will terminate.

This still all assumes that statements 1 and 2 are correct and so we will spend some time trying to prove them now.

Lemma 2.2. *Let $f_n(x) = x^n$ then $\forall i > n$ then*

$$D_i f_n(x) = 0$$

Proof. We prove this by induction on n. Let $n = 0$ then $f_0(x) = x^0 = 1$ so $D_1 f_0(x) = f_0(x+1) - f_0(x) = 1 - 1 = 0$. It is clear since $D_1 f_0(x) = 0$ that $D_j f_0(x) = 0$ for $j > 1$ also.

Assume that it is true for all positive integers up to and including n then we need to show it is true for $n + 1$.

We will show that $D_{n+2} f_{n+1}(x) = 0$ as it follows that $D_j f_{n+1}(x) = 0$ for $j > n + 2$

$$\begin{aligned}
D_{n+2} f_{n+1} = D_{n+1} D f_{n+1} &= D_{n+1}\big((x+1)^{n+1} - x^{n+1}\big) \\
&= D_{n+1}\Big(\sum_{i=0}^{n} \binom{n+1}{i} x^i\Big) \\
&= D_{n+1}\Big(\sum_{i=0}^{n} \binom{n+1}{i} f_i\Big) \\
&= \sum_{i=0}^{n} \binom{n+1}{i} D_{n+1} f_i \text{ \textit{(by corollary 2.1.1)}} \\
&= \sum_{i=0}^{n} \binom{n+1}{i} 0 \text{ \textit{(by the induction hypothesis)}} \\
&= 0
\end{aligned}$$

□

Corollary 2.2.1. *Let $f_n(x) = x^n$ then*

$$D_n f_n(x) = n!$$

Proof. We prove this by induction on n. Let $n = 0$ then $D_0 f_0(x) = f_0(x) = x^0 = 1 = 0!$ so $D_0 f_0(x) = 0!$.

Assume that it is true for all positive integers less than n then we will show it is true for n.

$$
\begin{aligned}
D_n f_n &= D_{n-1}\left(D f_n\right) \\
&= D_{n-1}\left((x+1)^n - x^n\right) \\
&= D_{n-1}\left(\sum_{i=0}^{n-1}\binom{n}{i}x^i\right) \\
&= D_{n-1}\left(\sum_{i=0}^{n-1}\binom{n}{i}f_i\right) \\
&= \sum_{i=0}^{n-1}\binom{n}{i}D_{n-1}f_i \ \textit{(by corollary 2.1.1)} \\
&= \binom{n}{n-1}D_{n-1}f_{n-1} \ \textit{(by lemma 2.2)} \\
&= \binom{n}{n-1}(n-1)! \ \textit{(by the induction hypothesis)} \\
&= n!
\end{aligned}
$$

□

Theorem 2.3. *Let* $f(x) = \sum_{i=0}^{n} c_i x^i$ *then* $D_n f(x) = c_n n!$

Proof.

$$
\begin{aligned}
D_n f &= D_n\left(\sum_{i=0}^{n} c_i x^i\right) \\
&= \sum_{i=0}^{n} c_i D_n\left(x^i\right) \\
&= c_n D_n\left(x^n\right) \ \textit{(by lemma 2.2)} \\
&= c_n n! \ \textit{(by corollary 2.2.1)}
\end{aligned}
$$

□